NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

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Abstract

We investigate the form assumed by the uncertainty relation when the signal involved is real and the uncertainty in frequency is defined as the moment of inertia about its centroid of the positive frequency power spectrum of the signal. A study of the general signal with a positive frequency spectrum obtained by suppressing the negative frequencies in an arbitrary Gaussian function shows that the uncertainty product can be appreciably less than one-half.

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1. Introduction

The purpose of the present note is to supplement in certain respects some work by Gabor [1] and Wolf [2] on the uncertainty relation for real signals. We begin by summarizing briefly the results obtained by these authors.

Let $f(t)$ be a function, in general complex, of the real variable t, and let $F(\omega)$ be its Fourier transform or spectrum, so that $f(t)$ and $F(\omega)$ are related by

$$(1) \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega,$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt.$$

If we assume (as we can without loss of generality) that

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = 1,$$

i.e., that $f(t)$ is normalized in the L^2-norm, then it follows by Parseval's theorem that

$$\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega = 1.$$

Thus $|f(t)|^2$ and $|F(\omega)|^2$ can be regarded as the density functions of distributions of unit mass on the infinite real line $(-\infty, \infty)$. The first and second moments of these distributions are the quantities

$$(2) \qquad < t > = \int_{-\infty}^{\infty} t|f(t)|^2 dt, \qquad < t^2 > = \int_{-\infty}^{\infty} t^2|f(t)|^2 dt,$$

$$< \omega > = \int_{-\infty}^{\infty} \omega|F(\omega)|^2 d\omega, \qquad < \omega^2 > = \int_{-\infty}^{\infty} \omega^2|F(\omega)|^2 d\omega ,$$

which will always be assumed to exist, and the <u>centered</u> second moments or <u>variances</u> are the quantities

$$(\triangle t)^2 = \int_{-\infty}^{\infty} (t - <t>)^2 \, |f(t)|^2 dt = <t^2> - <t>^2 ,$$

$$(\triangle \omega)^2 = \int_{-\infty}^{\infty} (\omega - <\omega>)^2 \, |F(\omega)|^2 d\omega = <\omega^2> - <\omega>^2 .$$

$\triangle t$ is often called the <u>uncertainty in t</u>, and $\triangle\omega$ the <u>uncertainty in ω</u>. It is well known that the <u>uncertainty product</u> $\triangle t \, \triangle\omega$ obeys the <u>uncertainty relation</u>

(3) $$\triangle t \, \triangle \omega \geq \frac{1}{2} ;$$

the equality sign holds only when f(t) and F(ω) have the special form

(4) $$f(t) = \frac{1}{\sqrt[4]{2\pi(\triangle t)^2}} \exp(i\gamma) \exp\left[- \frac{(t - <t>)^2}{4(\triangle t)^2} + i <\omega> t\right]$$

$$F(\omega) = \sqrt[4]{2(\triangle t)^2/\pi} \exp\left[i(\gamma + <t><\omega>)\right]$$

$$\exp\left[-(\triangle t)^2(\omega - <\omega>)^2 - i <t> \omega\right] ,$$

where γ is an arbitrary real number. Proof of these facts can be found in any textbook of quantum mechanics, a subject in which uncertainty relations play a fundamental role.

In an important class of applications, the functions f(t) are real, unlike the wave functions of quantum mechanics. For example, f(t) might be an electrical signal, an optical wave train, or the correlation function of a random noise process. The reality of f(t) implies the condition

$$F(\omega) = F^*(-\omega)$$

(where the asterisk denotes the complex conjugate), or equivalently

$$(5) \qquad |F(\omega)| = |F(-\omega)|,$$

$$\arg F(\omega) = -\arg F(-\omega),$$

so that the spectrum $F(\omega)$ is completely determined if it is given only for positive frequencies. It follows that for real signals the mean frequency $< \omega >$ defined by (2) is identically zero, and, in particular, that the equality in (3) can be achieved only for those of the functions (4) for which $< \omega > = \gamma = 0$. If, for example, $f(t)$ is a sine wave of frequency ω_o which is amplitude-modulated by a signal of bandwidth W, where $W << \omega_o$, the definition (2) compels us to choose zero as the mean frequency rather than the carrier frequency ω_o. To avoid this unnatural choice of mean frequency, Gabor replaced the signal $f(t)$ by the so-called <u>analytical</u> <u>signal</u>

$$f_+(t) = (1 / \sqrt{\pi}) \int_0^{\infty} F(\omega) \exp(i\omega t) d\omega.$$

The function $f_+(t)$ is complex, with real part $2^{-1/2} f(t)$ and imaginary part $2^{-1/2} h(t)$, where $h(t)$ is the Hilbert transform of $f(t)$. The Fourier transform of $f_+(t)$ is the function

$$(6) \qquad F_+(\omega) = \sqrt{2}\, F(\omega), \qquad \omega \geq 0,$$

$$F_+(\omega) = 0, \qquad \omega < 0,$$

which we shall call the <u>positive</u> <u>frequency</u> <u>spectrum</u> of $f(t)$. Again $|f_+(t)|^2$ and $|F_+(\omega)|^2$ can be regarded as the density functions of distributions of unit mass on the real line*; the moments corresponding to these distributions are

* Note that $\int_{-\infty}^{\infty} |f_+(t)|^2 dt = \int_{-\infty}^{\infty} |F_+(\omega)|^2 d\omega = 1.$

(7)
$$< t >_+ = \int_{-\infty}^{\infty} t|f_+(t)|^2 \, dt,$$

$$< t^2 >_+ = \int_{-\infty}^{\infty} t^2|f_+(t)|^2 dt,$$

$$(\triangle t_+)^2 = < t^2 >_+ - < t >_+^2 ,$$

$$< \omega >_+ = \int_{-\infty}^{\infty} \omega|F_+(\omega)|^2 d\omega = 2\int_0^{\infty} \omega|F(\omega)|^2 d\omega,$$

$$< \omega^2 >_+ = \int_{-\infty}^{\infty} \omega^2|F_+(\omega)|^2 d\omega = 2\int_0^{\infty} \omega^2|F(\omega)|^2 d\omega,$$

$$(\triangle\omega_+)^2 = < \omega^2 >_+ - < \omega >_+^2 .$$

It is easily seen that $(\triangle\omega_+)^2$ is the moment of inertia of the positive frequency spectrum of f(t) about its centroid $< \omega >_+$.

With these definitions it follows by the same argument that leads to (3) that

(8)
$$\triangle t_+ \triangle\omega_+ \geq \frac{1}{2} ,$$

except that the equality sign cannot be achieved, since all the functions (4) have negative frequencies in their spectra. One would now like to convert (8) into an uncertainty relation for the product $\triangle t \; \triangle\omega_+$, since $\triangle t$, rather than $\triangle t_+$, is the appropriate measure of the width of the given real signal f(t). How this is to be done is not indicated by Gabor. In the cited paper, Wolf studies the relation between $\triangle t$ and $\triangle t_+$, and shows that $\triangle t$ and $\triangle t_+$ are equal, if $\triangle t_+$ exists. ($\triangle t$ is always assumed to exist.) However, he finds that the existence of $\triangle t$ implies that of $\triangle t_+$ only if F(0) vanishes. If this condition is met, as it is in a large number of physical situations (in particular, the quasi-monochromatic optical wave trains with which Wolf is especially

concerned), then the uncertainty relation (8) can be replaced by the uncertainty relation

$$\Delta t\, \Delta\omega_+ \geq \frac{1}{2},$$

where the equality sign cannot be achieved, for the reason already mentioned.

In this note, we make two additions to the work of Gabor and Wolf. First, we develop a general inequality for the product $\Delta t\, \Delta\omega_+$ from which Wolf's result follows in the special case $F(0) = 0$. In doing so, we do not introduce the function $f_+(t)$ as an intermediate step. Second, we evaluate the uncertainty product $\Delta t \Delta\omega_+$ for a class of signals, whose positive frequency spectra are obtained by suppressing the negative frequencies in an arbitrary Gaussian function

$$g(m,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-(\omega-m)^2/2\,\sigma^2\right],$$

and then renormalizing to unity. It is found that the product $\Delta t\Delta\omega_+$ corresponding to functions of this type is always less than 1/2, and becomes approximately 0.3 for a suitable value of the ratio m/σ. Thus, the least value of the uncertainty product $\Delta t\Delta\omega_+$ is not 1/2, as sometimes stated in the literature; however, the problem of finding the least value (or greatest lower bound) of $\Delta t\Delta\omega_+$ remains unsolved.

2. An inequality for the product $\Delta t\Delta\omega_+$

In this section we derive an inequality for the uncertainty product $\Delta t\Delta\omega_+$ by suitably modifying the standard Pauli-Weyl proof of the relation (3) [see Weyl[3]]. As noted above, we completely avoid the use of the analytical signal $f_+(t)$, which in our view adds spurious difficulties to the problem. The details are as follows.

Since $tf(t)$ and $iF'(\omega)$ are Fourier transform pairs*, it follows from

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* The prime will be used to denote differentiation with respect to ω.

Parseval's theorem that

$$< t^2 > = \int_{-\infty}^{\infty} |F'(\omega)|^2 \, d\omega.$$

Moreover, writing $F(\omega)$ as*

$$F = |F| \exp(i \arg F),$$

we see from (5) that the quantity

$$|F'|^2 = (|F|')^2 + |F|^2 \left[(\arg F)'\right]^2$$

is even. Thus $< t^2 >$ can be written as

$$< t^2 > = 2 \int_{0}^{\infty} |F'(\omega)|^2 \, d\omega,$$

i.e., as an integral over the semi-infinite interval $(0,\infty)$, and the variance $(\triangle t)^2$ can be written as

(9) $$(\triangle t)^2 = 2 \int_{0}^{\infty} |F'(\omega)|^2 d\omega - < t >^2 .$$

To eliminate the second term in the right side of (9) we translate $f(t)$ to the left by an amount $< t >$ by introducing the function

(10) $$\tilde{f}(t) = f(t + < t >),$$

with Fourier transform

(11) $$\tilde{F}(\omega) = F(\omega) \exp(i < t > \omega).$$

It follows that

$$|F'|^2 = |\tilde{F}'|^2 + < t >^2 |\tilde{F}|^2 - i < t > \tilde{F} \tilde{F}'^{*} + i < t > \tilde{F}^{*}\tilde{F}' ,$$

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* For simplicity of notation, we often omit the arguments of the functions F, F', $|F|$, etc.

whence we easily obtain

(12) $$(\Delta t)^2 = \int_{-\infty}^{\infty} (|\tilde{F}'|^2 + < t >^2 |\tilde{F}|^2) d\omega - < t >^2 = \int_{-\infty}^{\infty} |\tilde{F}'|^2 d\omega .$$

In (12) we have used the fact that

$$\int_{-\infty}^{\infty} |\tilde{F}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = 1,$$

and the fact that the integrals

$$- i \int_{-\infty}^{\infty} \tilde{F}(\omega)\tilde{F}'^{*}(\omega) = i \int_{-\infty}^{\infty} \tilde{F}^{*}(\omega)\tilde{F}'(\omega) d\omega$$

are just the centroid of the distribution $|\tilde{f}(t)|^2$, and therefore vanish by construction. Since $|\tilde{F}'|^2$ is even, we can write

$$(\Delta t)^2 = 2 \int_{0}^{\infty} |\tilde{F}'(\omega)|^2 \, d\omega.$$

To express the variance $(\Delta\omega_+)^2$ in terms of $\tilde{F}(\omega)$, we use (7) and (11) and find that

(13) $$(\Delta\omega_+)^2 = 2 \int_{0}^{\infty} \omega^2 |\tilde{F}(\omega)|^2 d\omega - < \omega >_+^2 .$$

To eliminate the second term in the right side of (13), we translate $\tilde{F}(\omega)$ to the left by an amount $< \omega >_+$ by introducing the function

(14) $$\hat{F}(\omega) = \tilde{F}(\omega + < \omega >_+).$$

It is easily verified that in terms of $\hat{F}(\omega)$, the variance $(\Delta\omega_+)^2$ becomes

$$(\Delta\omega_+)^2 = 2 \int_{-< \omega >_+}^{\infty} \omega^2 \, |\hat{F}(\omega)|^2 d\omega,$$

and the square of the uncertainty produce $\Delta t \Delta\omega_+$ becomes

$$(15)\qquad (\triangle t)^2(\triangle\omega_+)^2 = 4\int_{-<\omega>_+}^{\infty} |\hat{F}'(\omega)|^2 d\omega \int_{-<\omega>_+}^{\infty} \omega^2 |\hat{F}(\omega)|^2 d\omega.$$

We now develop an inequality from (15) by using the Schwartz inequality in the form

$$(16)\qquad \int |x(\omega)|^2 d\omega \int |y(\omega)|^2 d\omega \geq \left[\int \mathrm{Re}\left\{x(\omega)y^*(\omega)\right\} d\omega\right]^2$$

$$= \left[\int \mathrm{Re}\left\{x^*(\omega)y(\omega)\right\} d\omega\right]^2,$$

where $x(\omega)$, $y(\omega)$ are any square-integrable functions, and the integrals are over any interval. Letting $\hat{F}'(\omega)$ be the function $x(\omega)$ in (16) and $\omega\hat{F}(\omega)$ the function $y(\omega)$, we obtain the inequality

$$(17)\qquad (\triangle t)^2(\triangle\omega_+)^2 \geq 4\left[\frac{1}{2}\int_{-<\omega>_+}^{\infty} \omega\left\{\hat{F}'(\omega)\,\hat{F}^*(\omega) + \hat{F}'^*(\omega)\,\hat{F}(\omega)\right\} d\omega\right]^2.$$

The right side of (17) is equal to

$$\left[\int_{-<\omega>_+}^{\infty} \omega \frac{d}{d\omega} |\hat{F}(\omega)|^2 d\omega\right]^2,$$

which becomes

$$\left[<\omega>_+ |\hat{F}(-<\omega>_+)|^2 - \int_{<\omega>_+}^{\infty} |\hat{F}(\omega)|^2 d\omega\right]^2$$

after integrating by parts. Finally using (10) and (14) to return to the original spectrum $F(\omega)$, we obtain the uncertainty relation

$$(18)\qquad \triangle t\triangle\omega_+ \geq \frac{1}{2}\left|1 - 2|F(0)|^2 <\omega>_+\right| = \frac{1}{2}\left|1 - |F_+(0)|^2 <\omega>_+\right|$$

[see (6)].

In the uncertainty relation (18), the equality cannot be achieved, unlike the case of the usual uncertainty relation (3). To see this, we note that the equality sign holds in (16) only when the functions $x(\omega)$ and $y(\omega)$ are proportional, i.e., only when

$$(19) \qquad \hat{F}'(\omega) = \lambda\omega\,\hat{F}(\omega), \qquad -<\omega>_+ \leq \omega < \infty,$$

where λ is a real constant. The square-integrable solutions of (19) are of the form

$$\hat{F}(\omega) = \alpha \exp(-\beta\,\omega^2), \qquad -<\omega>_+ \leq \omega < \infty,$$

where α and β are arbitrary real constants. Therefore, using (10) and (14), we find that in terms of the original function $F(\omega)$, the equality sign holds only for the functions

$$(20) \qquad F(\omega) = \alpha \exp\left[-i <t> \omega - \beta(\omega - <\omega>_+)^2\right], \quad 0 \leq \omega < \infty.$$

However, the functions (20) do not satisfy the constraint

$$\int_0^\infty \omega\,|F(\omega)|^2\,d\omega \Big/ \int_0^\infty |F(\omega)|^2 d\omega = <\omega>_+ ,$$

and are therefore not admissible spectra. This is obvious, since the centroid and maximum of a truncated Gaussian cannot coincide*. It follows that there is no choice of $F(\omega)$ for which the equality sign in (18) holds.

If $F(0) = 0$, then (18) reduces to the uncertainty relation

$$(21) \qquad \Delta t \Delta\omega_+ \geq \frac{1}{2}$$

found by Wolf (see Introduction). Moreover, if the quantity

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* In the usual uncertainty relation, this difficulty does not arise, since in the infinite interval the maximum and centroid of a Gaussian distribution coincide.

(22) $$\eta = 2|F(0)|^2 < \omega >_+ = |F_+(0)|^2 < \omega >_+$$

is less than δ, where $0 < \delta < 1$, then (18) reduces to

(23) $$\triangle t \triangle \omega_+ \geq \frac{1}{2}(1-\delta).$$

Thus, the smallness of η implies something like the familiar uncertainty relation (3). Although, as the reader can easily convince himself, the quantity η is small for many signals for which $\triangle\omega_+ << <\omega>_+$, i.e., for many signals which are usually classified as being narrow pass band signals, it is of course possible for η to be large even if $\triangle\omega_+ << <\omega>_+$. With this reservation, one can use the uncertainty principle (21) for narrow pass band signals. Again we note that in (21) and (23), the equality sign cannot hold; however, in the next section, we shall show when η is small we can come close to achieving the equality in (21) and (23).

3. The product $\triangle t \triangle \omega_+$ for a class of Gaussian spectra

In this section we evaluate the uncertainty product $\triangle t \triangle \omega_+$ for a simple class of functions G, namely those with positive frequency spectra obtained by cutting off the negative frequencies of an arbitrary Gaussian function

$$g(m,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-(\omega-m)^2/2\,\sigma^2\right] \quad (-\infty < m < \infty,\ 0 \leq \sigma < \infty),$$

and then renormalizing to unity. Two important points emerge from a study of the values of the product $\triangle t \triangle \omega_+$ for functions of the class G. First of all, it is found that $\triangle t \triangle \omega_+$ can become substantially less than 1/2, a question which (18) leaves open (since the equality sign cannot obtain). Secondly, and in the same vein, it is found that when the quality η defined by (22) is small,

$\Delta t \Delta\omega_+$ is very close to 1/2 for suitable $f(t) \varepsilon G$. In this sense, the uncertainty relation for narrow pass band signals might be written in the physically useful form

$$(24) \qquad \Delta t \Delta\omega_+ \gtrsim \frac{1}{2} ,$$

where <u>approximate</u> equality is possible.

To evaluate $\Delta t \Delta\omega_+$ for $f(t) \varepsilon G$ we note that (18) can be generalized to

$$(25) \qquad \Delta t \Delta\omega_m \geq \frac{1}{2} \left| 1 - 2|F(0)|^2 m \right| ,$$

where

$$(\Delta\omega_m)^2 = \int_{-\infty}^{\infty} \omega^2 |F_+(\omega)|^2 d\omega - m^2 = 2 \int_0^{\infty} \omega^2 |F(\omega)|^2 d\omega - m^2$$

is the moment of inertia of $|F_+(\omega)|^2$ about the line $\omega = m$. To see this, one need only parallel the derivation of (18) step by step, using $\Delta\omega_m$ instead of $\Delta\omega_+$ and m instead of $< \omega >_+$. Similarly, it follows that the equality sign in (25) holds only for functions with squares of the form*

$$(26) \qquad F^2(\omega) = N(m,\sigma) \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2} (\omega - m)^2 \right] \qquad \omega \geq 0$$

$$F^2(\omega) = N(m,\sigma) \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2} (\omega + m)^2 \right] \qquad \omega \leq 0,$$

i.e., for the spectra of the signals in the class G. The function $N(m,\sigma)$ which normalizes the functions (26) is easily found to be

$$(27) \qquad N(m,\sigma) = \frac{1}{2\,\Phi(\rho)} , \qquad -\infty < \rho < \infty,$$

where ρ is the ratio m/σ, and

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* For simplicity, we consider only signals centered at $< t >$.

$$\Phi(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\rho} e^{-u^2/2}\, du$$

is the distribution function of a Gaussian random variable with zero mean and unit variance. Thus we have shown that

$$(28) \qquad \Delta t \Delta\omega_m = \frac{1}{2}\left|1 - 2|F(0)|^2 m\right| = \frac{1}{2}\left[1 - \frac{\rho\, e^{-\rho^2/2}}{\sqrt{2\pi}\,\Phi(\rho)}\right], \qquad f(t) \in G.$$

To find $\Delta t \Delta\omega_+$ for the signals in G, we resort to the parallel axis theorem of mechanics, one form of which states that

$$(29) \qquad (\Delta\omega_+)^2 = (\Delta\omega_m)^2 - (m - <\omega>_+)^2$$

for a unit mass distribution. Solving (28) for Δt and using (29) we find that

$$(30) \qquad (\Delta t)^2(\Delta\omega_+)^2 = \frac{1}{4}\left[1 - \frac{\rho\, e^{-\rho^2/2}}{\sqrt{2\pi}\,\Phi(\rho)}\right]^2 \left[1 - \frac{(m - <\omega>_+)^2}{(\Delta\omega_m)^2}\right].$$

The quantities $<\omega>_+$ and $\Delta\omega_m$ are easily evaluated for functions in G, and are found to be

$$<\omega>_+ = m + \frac{\sigma\, e^{-\rho^2/2}}{\sqrt{2\pi}\,\Phi(\rho)},$$

$$(\Delta\omega_m)^2 = \sigma^2\left[1 - \frac{\rho\, e^{-\rho^2/2}}{\sqrt{2\pi}\,\Phi(\rho)}\right], \qquad f(t) \in G.$$

Substituting in (30), we obtain finally

$$(31) \qquad (\Delta t)^2(\Delta\omega_+)^2 = \frac{1}{4}\left[1 - \frac{\rho\, e^{-\rho^2/2}}{\sqrt{2\pi}\,\Phi(\rho)}\right]\left[1 - \frac{\rho\, e^{-\rho^2/2}}{\sqrt{2\pi}\,\Phi(\rho)} - \frac{e^{-\rho^2}}{2\pi\,\Phi^2(\rho)}\right] \equiv \frac{1}{4}\, U(\rho).$$

(It is perhaps of some interest to note that $U(\rho)$ can be written as

$$U(\rho) = \left[1 + \Phi''(\rho) / \Phi(\rho)\right]\left[1 + (\ln \Phi(\rho))''\right],$$

where the prime denotes differentiation with respect to ρ.) Using the asymptotic expansion of $\Phi(\rho)$ given by Feller [4], it is easy to show that $\lim_{\rho \to -\infty} U(\rho) = \lim_{\rho \to \infty} U(\rho) = 1$. Large positive values of ρ obviously correspond to small values of the quantity η defined in (22), so that as alleged at the beginning of this section, when η is small, the uncertainty product $\Delta t \Delta \omega_+$ is very close to 1/2 for a suitable function in G. The function $U(\rho)$ is sketched in Figure 1. It can be seen that $U(\rho)$ is always less than 1, and reaches a minimum value of about 1/3 for a value of ρ between 0.2 and 0.3. Thus, for a suitable $f(t) \varepsilon G$, the product $\Delta t \Delta \omega_+$ can be made as small as about 0.3, as stated in the Introduction.

4. Conclusion

The version of the uncertainty relation for real signals given by (18) is unsatisfactory in one respect. We know, of course, that the products $\Delta t \Delta \omega_+$ have a greatest lower bound, since they are bounded below by zero, but we have not shown that this g.l.b. is itself greater than zero. It therefore remains an open question whether or not the products $\Delta t \Delta \omega_+$ have a positive g.l.b. If the g.l.b. is zero, a possibility which seems to go against one's physical intuition, there would exist a sequence of signals for which the uncertainty products zpproach zero. By placing a restriction on the size of $|F_+(0)|^2 < \omega >_+$ we can guarantee a positive g.l.b. for $\Delta t \Delta \omega_+$, but it is not entirely clear what physical significance is to be ascribed to this restriction.

The authors wish to express their gratitude to Dr. E. Wolf for helpful discussions concerning the work reported here.

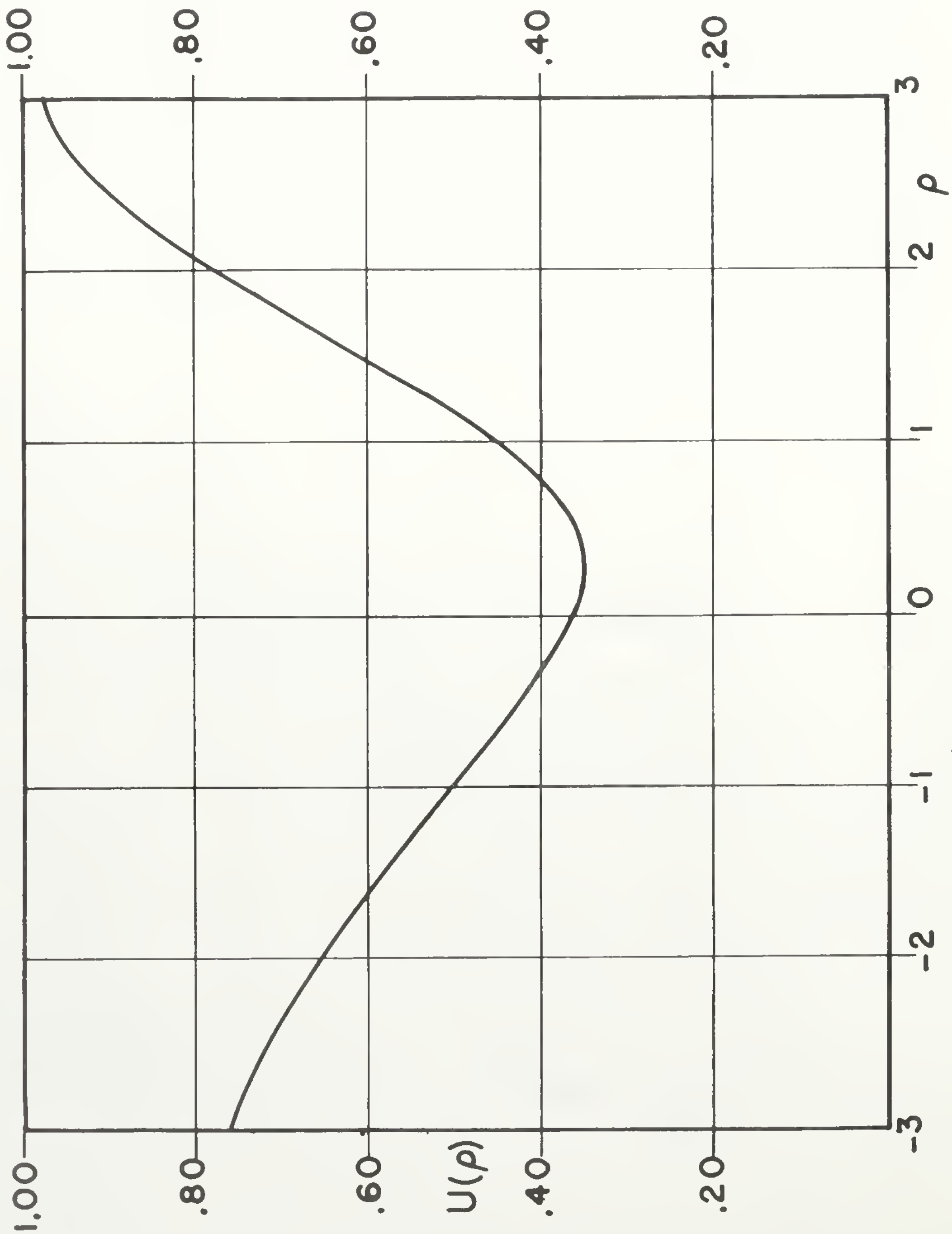

Figure 1

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